THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2014–2015)
Introduction to Topology
Exercise 5 Convergence

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let (x_n) be a sequence in (X, d) such that $d(x_n, x) \to c \in \mathbb{R}$ for a point $x \in X$. Can we conclude the convergence of (x_n)?

2. Given a sequence (x_n) and A be the set of points $\{x_n\}$.
   (a) Give an example of (x_n) that it converges and $\overline{A} \neq A$.
   (b) If $\overline{A} = A$, can you conclude anything about the convergence of (x_n)? Justify your conclusion by proof or examples.

3. Formulate a statement about the convergence of a sequence in $X \times Y$ (with product topology) with reference to the convergence of sequences in X and Y.

4. Let (X, d) be a metric space and two sequences in X satisfy $x_n \to x$ and $y_n \to y$. Show that $d(x_n, y_n) \to d(x, y)$.

5. Let (X, d) be a metric space. Show that if a sequence $x_n \to x$ then every subsequence of it converges to x. Show also the converse that if every convergent subsequence of (x_n) converges to x then $x_n \to x$. Is it true for general topological spaces.

6. Let X be a first countable space. Show that $x \in \overline{A}$ if and only if there is a sequence (a_n) in A converging to x. Moreover, show that $f\colon X \to Y$ is continous at $x_0 \in X$ if and only if for all sequence (x_n) converging to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.

7. Let $\mathbb{R}_{\ell\ell}$, $\mathbb{R}_{cf}$ and $\mathbb{R}$ be the real line with lower limit topology, cofinite topology, and standard topology respectively. Find examples of sequences that converge in one topology but not in another.